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CONTINUITY AND kTH ORDER DIFFERENTIABILITY IN ORLICZ–SOBOLEV SPACES: W^kL_A

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ABSTRACT
The paper gives a necessary and sufficient condition for the embedding of the Orlicz–Sobolev space $W^kL_A(\Omega)$ in $C(\Omega)$. The same condition is also found to be necessary and sufficient so that a continuous function in $W^kL_A(\Omega)$ be differentiable of order k almost everywhere in Ω.

Introduction

Let Ω be a bounded domain in R^n. The Orlicz–Sobolev space $W^kL_A(\Omega)$ is the set of all functions u in the Orlicz space $L_A(\Omega)$ such that the distributional derivatives $D^\alpha u$ are contained in $L_A(\Omega)$ for all α with $|\alpha| \leqq m$. For the definition and basic properties of Orlicz spaces, the reader is referred to [5]. The notation in this paper follows the one in [2].

The first result of this paper concerns conditions for the embedding of $W^kL_A(\Omega)$ in $C(\Omega)$, the space of continuous functions in Ω. E. A. Rozenfel'd showed in [8] that a necessary and sufficient condition for $W^1L_A(\Omega)$ to be embedded in $C(\Omega)$ is that

$$\int_0^1 \bar{A}(t^{1-n})t^{n-1}dt < \infty.$$

We extend the above result to the case $k > 1$: the necessary and sufficient condition for the embedding mentioned above to hold is:

$$\int_0^1 \bar{A}(t^{k-n})t^{n-1}dt < \infty.$$

This result sharpens a theorem due to Donaldson and Trudinger [2, theor.

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3.9(b)] in which they gave a sufficient (but not necessary) condition for the embedding of $W^k L_A(\Omega)$ in $C(\Omega)$.

The second result shows that a condition equivalent to the one that appears in the previous result, namely,

$$\int_1^\infty \left[\frac{t}{A(t)}\right]^{k/(n-k)} dt < \infty$$

is necessary and sufficient for a continuous function in $W^k L_A$ to have a differential of order k a.e. This was proved for $k = 1$ by A. P. Calderón in [1]. Our proof for $k > 1$ is in part a generalization of Calderón's proof. In the case where $A(t) = |t|^p$, the existence of a kth order differential a.e. was proved by Reshetnjak [7].

The results of this paper form part of the author's M. Sc. thesis. The author wishes to express his gratitude to Professor M. Marcus for his guidance and to Professor P. Saphar for his interest in this work.

1.

The embedding result can be stated as:

THEOREM 1. *Let $\Omega \subset R^n$ be a bounded open set with the cone property (see following lemma for definition). If A is an N-function such that:*

$$\int_0^1 \bar{A}(t^{k-n})\, t^{n-1} dt < \infty \qquad \text{then:} \tag{1.1}$$

i) *$W^k L_A(\Omega)$ can be continuously embedded in $C(\Omega)$,*

ii) *for each $f \in W^k L_A(\Omega)$, there exists a continuous $\tilde{f}$ such that $\tilde{f}(x) = f(x)$ for each Lebesgue point of f, and therefore $\tilde{f} = f$ a.e.,*

iii) $\sup_{x \in \Omega} |\tilde{f}(x)| \leqq c \cdot \|f\|_{W^k L_A}$.

The proof of the theorem makes use of the following two lemmas. A result similar to the one in the first of these can be found in [6]; our notation follows that of [6].

LEMMA 1.1. *Let $u \in C_c^\infty(\Omega)$, where $\Omega \subset R^n$ is an open set which has the cone property, that is for each x, there exists a cone*

$$\Gamma_{\sigma_x} = \{y : y = t\sigma,\ 0 \leqq t < T \leqq \infty,\ |\sigma - \sigma_x| < \gamma;\ |\sigma| = |\sigma_x| = 1\}$$

such that $x + \Gamma_{\sigma_x} \subset \Omega$. ($\gamma$ and T depend only on Ω).
Then for $x \in \tilde{K} = \operatorname{supp} u$, u can be represented as

$$u(x)=\sum_{|\alpha|=k}\int_{\Gamma_{\sigma_x}\cap(\tilde{K}-x)} D^\alpha u(x+y)\frac{y^\alpha}{|y|^n}h_x\left(\frac{y}{|y|}\right)dy$$

where $h_x(\sigma)=h(\sigma-\sigma_x)$ *and* $h\in C_c^\infty$ *is defined on a set isomorphic to a subset of* R^{n-1}, $\operatorname{supp} h\subset\{\sigma:|\sigma|<\gamma\}$ *and such that:*

$$\int_{|\sigma|<\gamma} h(\sigma)d\sigma=[(-1)^k\cdot(k-1)!]^{-1}.$$

PROOF. Since $u\in C_c^\infty(\Omega)$, for $x\in\tilde{K}$ and $|\sigma-\sigma_x|<\gamma$:

$$\int_0^\infty t^{k-1}\frac{d^k}{dt^k}u(x+t\sigma)dt=(-1)^k(k-1)!\,u(x).$$

After multiplying both sides by h_x and integrating over $|\sigma-\sigma_x|<\gamma$, we get:

$$\int_{|\sigma-\sigma_x|<\gamma} h_x(\sigma)\int_0^\infty t^{k-1}\frac{d^k}{dt^k}u(x+t\sigma)dt=u(x).$$

Now, by the change of variable $y=t\cdot\sigma$ and keeping in mind the fact that:

$$D^\alpha u(x+y)\neq 0\Rightarrow y\in\tilde{K}-x$$

we get the desired result. Q.E.D.

LEMMA 1.2. *Let* $u\in L_A(\Omega)$, Ω *a bounded set:*

i) *If* $u\in L_A'(\Omega)=\{u\mid\int_\Omega A(u(x))dx<\infty\}$, $K\subset\Omega$ *is compact, then:* $\forall\varepsilon>0\,\exists\delta(\varepsilon)$ *such that* $\int_K A(\frac{1}{3}[u(x+h)-u(x)])<\varepsilon\ \forall|h|<\delta$.

ii) *If* $u_\varepsilon=u*j_\varepsilon$ *is the regularization of* u, *then* $u_\varepsilon\to u$ *in the weak * topology* $\sigma(L_A,L_{\bar{A}})$ *as* $\varepsilon\to 0$.

REMARK. Part (ii) was proved in [4]. However it can be obtained also as a consequence of part (i).

PROOF.

i) If v is bounded $\int_K A(v(x+h)-v(x))dx\to 0$ as $|h|\to 0$. Now each $u\in L_A'$ can be written as $u=v+g$ where v is bounded and $\int_\Omega A(g(x))dx<\varepsilon/3$ for a given $\varepsilon>0$. Therefore by convexity:

$$\int_K A\left(\frac{u(x+h)-u(x)}{3}\right)dx\leqq\tfrac{1}{3}\int_K A(v(x+h)-v(x))dx+\tfrac{1}{3}\int_K A(g(x+h))dx$$

$$+\tfrac{1}{3}\int_K A(g(x))dx<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

ii) Given $w\in L_{\bar{A}}$ and $\rho>0$, by Young's inequality we have:

$$\left|\int_K [u(x+h)-u(x))]\,w(x)dx\right| \leqq \rho\int_K A\left(\frac{u(x+h)-u(x)}{3}\right)dx + \rho\int_K \bar{A}\left(\frac{3w(x)}{\rho}\right)dx.$$

Both terms on the right hand side tend to 0, the first one because of (i) and the second one by the definition of an N function. The assertion is a consequence of the above inequality using arguments similar to the one in the proof of Theorem 1 below.

Proof of Theorem 1. For $k > n$, the result follows from Sobolev's embedding theorem and the fact that $W^kL_A \subset W_{k,1}$, this embedding being continuous. Therefore let $k \leqq n$ (see [3]).

Let $f \in W^kL_A(\Omega)$, $K \subset \Omega$ compact and $0 < \varepsilon_0 < \operatorname{dist}(K, \partial\Omega)$; there exists a function $\eta \in C^\infty(\Omega)$ such that:

$$\eta \equiv 1 \quad \text{on} \quad K_{\varepsilon_0} = \{z \mid \exists y \in K : |z-y| \leqq \varepsilon_0\} \quad \text{and} \quad \operatorname{supp}\eta = K_1 \subset \Omega.$$

Denote $g = f\cdot\eta$; then $D^\alpha g \in L_A$ for $|\alpha| \leqq k$. In order to prove (i) we shall show that g is equivalent to a continuous function defined on K.

Let $0 < \varepsilon_1$, $\varepsilon_2 < \operatorname{dist}(K_1, \partial\Omega)$, $x \in K$; let us define the regularizations $g_{\varepsilon_i} = g * j_{\varepsilon_i}$, $i = 1, 2$. By using Lemma 1.1 for $g_{\varepsilon_1} - g_{\varepsilon_2}$ which has its support in a compact set $\tilde{K} \subset \Omega$, and using the fact that h has bounded support:

$$(1.2)\qquad |g_{\varepsilon_1}(x) - g_{\varepsilon_2}(x)| \leqq C\cdot\sum_{|\alpha|=k}\int_{\Gamma_{\sigma_x}\cap(\tilde{K}-x)} |D^\alpha_{g_{\varepsilon_1}}(x+y) - D^\alpha_{g_{\varepsilon_2}}(x+y)|\,|y|^{k-n}dy$$

$$\leqq c\cdot\sum_{|\alpha|=k}\sum_{i=1}^{2}\int_{\Gamma_{\sigma_x}\cap(\tilde{K}-x)} |D^\alpha_{g_{\varepsilon_i}}(x+y) - D^\alpha_g(x+y)|\,|y|^{k-n}dy$$

$$\leqq c\cdot\sum_{|\alpha|=k}\sum_{i=1}^{2}\int_{\tilde{K}} |D^\alpha_{g_{\varepsilon_i}}(z) - D^\alpha_g(z)|\,|z-x|^{k-n}dz.$$

Now since $D^\alpha_{g_{\varepsilon_i}} = (D^\alpha_g)_{\varepsilon_i}$

$$\int_{\tilde{K}} |D^\alpha_{g_{\varepsilon_i}}(z) - D^\alpha_g(z)|\,|z-x|^{k-n}dz$$

$$\leqq \int_{\tilde{K}}\int_\Omega |D^\alpha g(z-u) - D^\alpha g(z)|\,j_{\varepsilon_i}(u)du\cdot|z-x|^{k-n}dz$$

$$= \int_\Omega\left(\int_{\tilde{K}} |D^\alpha g(z-u) - D^\alpha g(z)|\,|z-x|^{k-n}dz\right)j_{\varepsilon_i}(u)du.$$

Now let $|u| < \varepsilon_i$, and let us consider the inner integral; by Young's inequality (for $\rho > 0$):

$$\begin{aligned}\int_{\tilde K} |D^\alpha g(z-u) - D^\alpha g(z)| \, |z-x|^{k-n} dz \\ = 3\rho \int_{\tilde K} \frac{|D^\alpha g(z-u) - D^\alpha g(z)|}{3} \frac{|z-x|^{k-n}}{\rho} dz \\ \leqq 3\rho \int_{\tilde K} A\left(\frac{D^\alpha g(z-u) - D^\alpha g(z)}{3}\right) dz + 3\rho \int_{\tilde K} \bar A\left(\frac{|z-x|^{k-n}}{\rho}\right) dz.\end{aligned}$$

Let $\varepsilon > 0$ be given, then we can find a $\delta > 0$ such that $\int_0^\delta \bar A\,(t^{k-n}) t^{n-1} dt < \varepsilon / 12\omega_{n-1}$ where ω_{n-1} is the surface area of the n-dimensional unit ball. Since $\bar A$ is an N-function for this $\delta > 0$ we can find a $\rho > 1$ such that

$$\rho \bar A\left(\frac{\delta^{k-n}}{\rho}\right) < \frac{\varepsilon}{12\omega_{n-1} R^n}$$

where$R > 0$ (which depends on K) is such that $\tilde K - K \subseteq B(0, R)$; and thus by the convexity of $\bar A$:

$$\begin{aligned}3\rho \int_{\tilde K} \bar A\left(\frac{|z-x|^{k-n}}{\rho}\right) dz \\ \leqq 3\rho\omega_{n-1} \int_0^\delta \bar A\left(\frac{t^{k-n}}{\rho}\right) t^{n-1} dt + 3\rho\omega_{n-1} \int_\delta^R \bar A\left(\frac{t^{k-n}}{\rho}\right) t^{n-1} dt \\ \leqq 3\omega_{n-1} \int_0^\delta \bar A\,(t^{k-n}) t^{n-1} dt + 3\rho\omega_{n-1} \bar A\left(\frac{\delta^{k-n}}{\rho}\right) R^n \\ < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.\end{aligned}$$

Now by use of Lemma 1.2 (i) we can find an $\varepsilon^* > 0$ such that for $|u| < \varepsilon^*$:

$$3\rho \int_{\tilde K} A\left(\frac{D^\alpha g(z-u) - D^\alpha g(z)}{3}\right) < \frac{\varepsilon}{2}.$$

So that on the whole for $0 < \varepsilon_1, \varepsilon_2 < \varepsilon^*$:

$$|g_{\varepsilon_1}(x) - g_{\varepsilon_2}(x)| < 2n^k c\varepsilon.$$

By taking the sup on K, we find that $\{g_\varepsilon\}$ is a Cauchy sequence in $C(K)$ and therefore there exists a continuous function $\tilde f$ such that $\tilde f = f$ a.e.

Now since $g_\varepsilon(x) \to g(x)$ for each x which is a Lebesgue point of f, we get (ii). By using Lemma 1.1 for g_ε we have as in (1.2)

$$|g_\varepsilon(x)| \leqq c \sum_{|\alpha|=k} \int_\Omega |D^\alpha g(z)| \, |z-x|^{k-n} dz.$$

Since $|z-x|^{k-n} \in L_{\bar A}$ by use of Lemma 1.2 (ii), letting $\varepsilon \to 0$, we get:

$$|\tilde{f}(x)| \leqq c \sum_{|\alpha|=k} \int_\Omega |D^\alpha g(z)| \, |z-x|^{k-n} dz$$

and by use of a known inequality in Orlicz spaces (see [5])

$$\leqq c \sum_{|\alpha|=k} \| D^\alpha g \|_A \max\left(1, \int_\Omega \bar{A}(|z-x|^{k-n}) dz\right)$$

and for bounded Ω we get:

$$\sup_{x \in \Omega} |\tilde{f}(x)| \leqq c_1 \| u \| \, W^k L_A(\Omega). \qquad \text{Q.E.D.}$$

2.

In this section we compare the embedding result of Donaldson and Trudinger [2, theor. 3.9(b)] with Theorem 1 of the present paper. We shall show that their condition for the continuous embedding $W^k L_A(\Omega) \hookrightarrow C(\Omega)$ implies that of Theorem 1 and that in fact there exist functions satisfying the condition of Theorem 1 but not that of [2].

We assume below that $k < n$; for $k \geqq n$ by Theorem 1, $W^k L_A(\Omega)$ can be continuously embedded in $C(\Omega)$. (See [3].)

First, we recall the condition given in [2]. For a given N-function A an integer $q(A)$ and a sequence of N-functions $C_0, C_1, \cdots, C_q$ are defined by the formulas

$$C_\nu^{-1}(x) = \int_0^x \frac{C_{\nu-1}^{-1}(t)}{t^{1+1/n}} dt$$

$$C_0(x) = A(x).$$

Assuming $\int_0^1 (C_\nu^{-1}(t)/t^{1+1/n}) dt < \infty$ for $\nu = 0, 1, \cdots, q$, $q(A) = q < n+1$, is such that

$$\int_0^\infty \frac{C_{q-1}^{-1}(t)}{t^{1+1/n}} dt = \infty \quad \text{but} \quad \int_0^\infty \frac{C_q^{-1}(t)\, dt}{t^{1+1/n}} < \infty. \tag{2.1}$$

The condition given in [2, theor. 3.9(b)] for the continuous embedding $W^k L_A(\Omega) \hookrightarrow C(\Omega)$ is:

$$k > q(A). \tag{2.2}$$

We show now that this condition implies condition (1.1) of Theorem 1,

$$\int_0^\infty \frac{C_q^{-1}(t)}{t^{1+1/n}}\,dt = \int_0^\infty \int_0^t \frac{C_{q-1}^{-1}(s)}{s^{1+1/n}}\,ds \frac{dt}{t^{1+1/n}}$$

$$(2.3)\qquad = \int_0^\infty \int_s^\infty \frac{dt}{t^{1+1/n}} \frac{C_{q-1}^{-1}(s)}{s^{1+1/n}}\,ds = n\int_0^\infty \frac{C_{q-1}^{-1}(s)}{s^{1+2/n}}\,ds$$

$$= n^q \int_0^\infty \frac{A^{-1}(s)}{s^{1+(q+1)/n}}\,ds$$

k and q being integers; $k > q \Rightarrow k \geqq q+1$. Therefore from (2.1) and (2.3) it follows that:

$$\int_0^\infty \frac{A^{-1}(s)}{s^{1+k/n}}\,ds < \infty$$

and by Young's inequality $s \leqq \bar{A}^{-1}(s)A^{-1}(s) \leqq 2s$ we get

$$(2.4)\qquad \int_0^\infty \frac{ds}{\bar{A}^{-1}(s)s^{k/n}} < \infty .$$

On the other hand

$$(2.5)\qquad \int_0^1 \bar{A}(t^{k-n})t^{n-1}\,dt = c\int_1^\infty \frac{\bar{A}(\tau)}{\tau^{2+k/(n-k)}}\,d\tau$$

$$= c\int_{A(1)}^\infty \frac{s}{[\bar{A}^{-1}(s)]^{2+k/(n-k)}}\,d(\bar{A}^{-1}(s)).$$

Since the integrand in (2.4) is decreasing, Abel's theorem implies that

$$\lim_{s\to\infty} \frac{s}{\bar{A}^{-1}(s)s^{k/n}} = 0 .$$

In particular, for large values of s:

$$s^{1-k/n} \leqq \bar{A}^{-1}(s).$$

Therefore for sufficiently large s_0:

$$(2.6)\qquad \int_{s_0}^\infty \frac{ds}{[\bar{A}^{-1}(s)]^{1+k/(n-k)}} \leqq \int_{s_0}^\infty \frac{ds}{\bar{A}^{-1}(s)s^{k/n}} < \infty .$$

In view of Abel's theorem (2.6) implies that

$$(2.7)\qquad \lim_{s\to\infty} \frac{s}{[\bar{A}^{-1}(s)]^{1+k/(n-k)}} = 0 .$$

Finally, (2.6) and (2.7) imply that the last integral in (2.5) converges. This can be seen by integration by parts. We have thus shown that (2.2) implies (1.1).

We shall now consider an N-function $A(t)$ such that for large values of t

$$A(t) = t^{n/k}(\log t)^{\gamma} \quad \text{where} \quad \frac{n-k}{k} < \gamma \leqq \frac{n}{k}.$$

We shall show that $A(t)$ satisfies (1.1) so that the embedding $W^k L_A(\Omega) \hookrightarrow C(\Omega)$ is continuous while the condition (2.2) of [2] is not satisfied by $A(t)$.

For A as above, the function $\bar{A}$ complementary to A is equivalent to the N-function which is equal to $t^{n/(n-k)}(\log t)^{\gamma k/(k-n)}$ for large values of t. (See [5, p. 65].) Without loss of generality, let us assume that $\bar{A}$ is equal to the above function. It follows that for sufficiently small t_0:

$$\int_0^{t_0} \bar{A}(t^{k-n})\, t^{n-1}\, dt = c \cdot \left[(\log t^{k-n})^{(\gamma \cdot k/(k-n))+1}\right]_0^{t_0}.$$

The above integral converges for $\gamma > (n-k)/k$. Thus A satisfies (1.1) for these values of γ.

However for the same values of γ, $q(A) \geqq k$ since

$$(2.8) \qquad \int_1^{\infty} \frac{A^{-1}(t)}{t^{1+k/n}}\, dt = \int_{A^{-1}(1)}^{\infty} \frac{s}{[A(s)]^{1+k/n}}\, d(A(s))$$

diverges. Indeed

$$\int_{s_0}^{\infty} \frac{ds}{[A(s)]^{k/n}} = \int_{s_0}^{\infty} \frac{ds}{s(\log s)^{\gamma \cdot k/n}}$$

diverges for $\gamma \leqq n/k$, and

$$\lim_{s \to \infty} \frac{s}{[A(s)]^{k/n}} = \lim_{s \to \infty} \frac{1}{(\log s)^{\gamma \cdot k/n}} = 0.$$

Thus, integrating by parts (2.8), we get:

$$\int_1^{\infty} \frac{C_{k-1}^{-1}(t)}{t^{1+1/n}}\, dt = \infty.$$

It has thus been shown that $q(A) \nless k$ and therefore in this case, the criterion in [2] cannot applied.

3.

The following lemma is similar to Lemma 1.1 and is proved in a similar way. We state it here without proof.

LEMMA 3.1. *If u is infinitely differentiable such that* $\operatorname{supp} u \subset \overline{B(x_0, R')}$ *then for each* $x \in B(x_0, R')$

(3.1) $$u(x) = \sum_{|\alpha|=k} \int_{\{y \mid x+y \in B(x_0, R')\}} D^\alpha u(x+y) \frac{y^\alpha}{|y|^n} h\left(\frac{y}{|y|}\right) dy$$

where $h \in C_c^\infty(S_{n-1})$ *and* $\int_{|\sigma|=1} h(\sigma) d\sigma = [(-1)^k (k-1)!]^{-1}$.

The next assertion is an extension to the case $k > 1$ of a lemma of Calderón [1].

LEMMA 3.2. *If A is an N-function such that* $\int_0^\infty \left[\frac{t}{A(t)}\right]^{k/(n-k)} dt < \infty$, *then*

(3.2) $$\int_K u(y) |y-x|^{k-n} dy \leqq C \cdot \left(\int_K A(u(y)) dy\right)^{k/n}$$

where $u \geqq 0$ *and K is measurable.*

PROOF. Let $E_m = \{y \in K \mid 2^m \leqq u(y) \leqq 2^{m+1}\}$.

Then:

$$\int_K u(y) |y-x|^{k-n} dy$$

$$\leqq \sum_{m=-\infty}^{\infty} 2^{m+1} \int_{E_m} |y-x|^{k-n} dy \leqq \sum_{m=-\infty}^{\infty} 2^{m+1} \int_{B(x,r)} |y-x|^{k-n} dy$$

where $r = (|E_m|/\omega)^{1/n}$, $|E_m|$ = measure of E_m, ω = volume of the unit ball

$$= n\omega \sum_{m=-\infty}^{\infty} 2^{m+1} \int_0^r \rho^{k-1} d\rho = \frac{2n}{k} \omega^{1-k/n} \sum_{m=-\infty}^{\infty} 2^m |E_m|^{k/n}$$

$$= c_1 \sum_{m=-\infty}^{\infty} 2^m |E_m|^{k/n} \frac{A(2^m)^{k/n}}{A(2^m)^{k/n}}$$

$$\leqq c_1 \left(\sum_{m=-\infty}^{\infty} |E_m| A(2^m)\right)^{k/n} \left(\sum_{m=-\infty}^{\infty} \left(\frac{2^m}{A(2^m)^{k/n}}\right)^{n/(n-k)}\right)^{(n-k)/n}$$

$$\leqq c_2 \left(\int_K A(u(y)) dy\right)^{k/n} \left(\sum_{m=-\infty}^{\infty} \left(\frac{2^{mn/k}}{A(2^m)}\right)^{k/(n-k)}\right)^{(n-k)/n}.$$

Now, since A is increasing:

$$\left[\frac{2^{m\,n/k}}{A(2^m)}\right]^{k/n-k} \leqq \int_{m-1}^{m} \left[\frac{2^{(s+1)n/k}}{A(2^s)}\right]^{k/(n-k)} ds$$

$$\sum_{m=-\infty}^{\infty} \left(\frac{2^{m\,n/k}}{A(2^m)}\right]^{k/(n-k)} \leqq \int_{-\infty}^{\infty} \left[\frac{2^{s\,n/k}\,2^{n/k}}{A(2^s)}\right]^{k/n-k} ds$$

$$= \frac{2^{n/(n-k)}}{\log 2} \int_0^\infty \left[\frac{t}{A(t)}\right]^{k/(n-k)} dt = c_2 < \infty$$

and from here the result follows.

By making use of a known lemma (see [9] for a more general version) and the fact that $(D^\alpha u)_\varepsilon = D^\alpha u_\varepsilon$ we have:

Lemma 3.3. *Let Ω be a bounded domain, and let $u \in W_{1,1}(\Omega)$ have distributional derivatives $D^\alpha u$ for $|\alpha| = k$. Let us denote:*

$$\Omega_{(\varepsilon)} = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

For $x \in \Omega_{(\varepsilon)}$, let $u_\varepsilon(x) = u * j_\varepsilon(x)$. If $A: R \to R^+$ is convex and continuous then:

i) $$\int_{\Omega_{(\varepsilon)}} A(D^\alpha u_\varepsilon(x))\,dx \leqq \int_\Omega A(D^\alpha u(x))\,dx \qquad \varepsilon > 0$$

ii) $$\lim_{\varepsilon \to 0} \int_{\Omega_{(\varepsilon)}} A(D^\alpha u_\varepsilon(x))\,dx = \int_\Omega A(D^\alpha u(x))\,dx.$$

Theorem 2. *Let $f \in L^1_{\text{loc}}(R^n)$ be a continuous function such that its distributional derivatives $D^\alpha f \in L_A(K)$ for $|\alpha| = k < n$ and for each compact $K \subset R^n$, where A is an N-functions for which*

$$\int_1^\infty \left[\frac{t}{A(t)}\right]^{k/(n-k)} dt < \infty. \tag{3.3}$$

Then f has a kth order differential a.e. in R^n, that is for almost every x:

$$f(x+h) - f(x) - \sum_{0<|\alpha|\leqq k} \frac{(D^\alpha f)(x)\,h^\alpha}{\alpha!} = o(|h|^k) \quad \text{as} \quad |h| \to 0.$$

Proof. Let $x \in R^n$, we shall show that f is differentiable of order k in x which is outside a set of measure zero which shall be fixed later.

First of all, without loss of generality we may suppose that $\int_0^\infty [t/A(t)]^{k/(n-k)} dt < \infty$, since otherwise we can take an equivalent N-function for which the above assumption holds.

Secondly, we suppose that:

$$\int_{B(x,2)} A(D^\alpha f(y))\,dy < \infty, \qquad |\alpha| = k$$

for otherwise we may consider

$$\tilde{f}_x = \frac{f}{\sum_{|\alpha|=k} \| D^\alpha f \|_{L_A(B(x,2))}}.$$

Now, our first aim will be to show:

$$|f(x)| \leqq c \cdot \sum_{|\alpha|=k} \left(\int_S A(D^\alpha f(y))\,dy \right)^{k/n} \tag{3.4}$$

where $S = B(x_0, R)$, $R < 1$ is a ball which contains x. (For discontinuous f, this inequality would be true only for those x which are Lebesgue points of f.)

Let us define a sequence of functions $\{f_m\}$ such that:

$$f = f_m \quad \text{on} \quad S_{(m)} = \{x \in S \mid \operatorname{dist}(x, \partial S) \geqq 1/m\}$$

and such that $\operatorname{supp} f_m \subset \operatorname{int} S_{(m+1)} \subset S$.

Let us denote $S_\varepsilon = \{x \mid \exists y \in S : |x - y| \leqq \varepsilon\}$ and define:

$$f_{m\varepsilon}(x) = j_\varepsilon^* f_m(x) = \int_{|x-y| \leqq \varepsilon} f_m(y) j_\varepsilon(x-y)\,dy$$

then $\operatorname{supp} f_{m\varepsilon} \subset (\operatorname{supp} f_m)_\varepsilon \subset (S_{(m+1)})_\varepsilon = B(x, R')$ where $R' = R - (1/(m+1)) + \varepsilon$. From (3.1):

$$|f_{m\varepsilon}(x)| \leqq c \cdot \sum_{|\alpha|=k} \int_{\{y \mid x+y \in \operatorname{supp} f_{m\varepsilon}\}} |D^\alpha f_{m\varepsilon}(x+y)|\,|y|^{k-n}\,dy$$

$$\leqq C \sum_{|\alpha|=k} \int_{(S_{(m+1)})_\varepsilon} |D^\alpha f_{m\varepsilon}(z)|\,|z - x|^{k-n}\,dz \tag{3.5}$$

$$\leqq C_1 \sum_{|\alpha|=k} \left(\int_{(S_{(m+1)})_\varepsilon} A\left(\frac{D^\alpha f_{m\varepsilon}(y)}{2} \right) dy \right)^{k/n} \text{ from (2.2).}$$

We shall now estimate the limit of the right hand side when $\varepsilon \to 0$ and $m \to \infty$. Let:

$$\int_{(S_{(m+1)})_\varepsilon} A\left(\frac{D^\alpha f_{m\varepsilon}(y)}{2} \right) dy = \int_{B(x_0, R-(1/(m+1))-\varepsilon)} + \int_{B(x_0, R-(1/(m+1))+\varepsilon) \setminus B(x_0, R-(1/(m+1))-\varepsilon)}$$

$$= I_1 + I_2.$$

For $f_\varepsilon = f^* j_\varepsilon$ and $z \in B(x_0, R-(1/(m+1))-\varepsilon)$, by definition:

$$|D(f_{m\varepsilon}-f_\varepsilon)(z)| \leqq \int_{|z-y|\leqq\varepsilon} |D_y^\alpha j_\varepsilon(z-y)|\,|f_m(y)-f(y)|\,dy$$

$$\leqq \frac{c}{\varepsilon^{|\alpha|}} \int_{S_{(m+1)}} |f_m(y)-f(y)|\,dy = c(\varepsilon)\int_{S_{(m+1)}\setminus S_{(m)}} |f_m(y)-f(y)|\,dy.$$

Therefore, $D^\alpha f_{m\varepsilon}(z) \leqq D^\alpha f_\varepsilon(z) + \delta(\varepsilon, m)$ where $\lim_{m\to\infty}\delta(\varepsilon,m)=0$.
So, by the convexity of A,

$$I_1 \leqq \tfrac{1}{2}\int_{B(x_0,R-(1/(m+1))-\varepsilon)} A(D^\alpha f_\varepsilon(z))\,dz + \tfrac{1}{2}A(\delta)\,|B(x_0,R)|$$

$$\leqq \tfrac{1}{2}\int_{B(x_0,R-\varepsilon)} A(D^\alpha f_\varepsilon(z))\,dz + \tfrac{1}{2}A(\delta)\,|B(x_0,R)|.$$

By Lemma 3.3, the first term on the right hand side is bounded by and converges to:

$$\tfrac{1}{2}\int_{B(x_0,R)} A(D^\alpha f(y))\,dy.$$

The second term obviously $\to 0$, when $m\to\infty$, so that

$$\lim_{m\to\infty} I_1 \leqq \tfrac{1}{2}\int_{B(x_0,R)} A(D^\alpha f(y))\,dy.$$

Let us consider now:

$$I_2 = \int_{P_\varepsilon} A\left(\frac{D^\alpha f_{m\varepsilon}(y)}{2}\right) dy$$

where $P_\varepsilon = \{R-(1/(m+1))-\varepsilon < |y-x_0| < R-(1/(m+1))+\varepsilon\}$ and let us choose Q such that:

$$Q_{(\varepsilon)} = \{x \in Q \mid \operatorname{dist}(x,\partial Q) > \varepsilon\} = P_\varepsilon$$

that is $Q = \{y \mid R-(1/(m+1))-2\varepsilon < |y-x_0| < R-(1/(m+1))+2\varepsilon\}$.
Then according to Lemma 3.3:

$$I_2 = \int_{Q_{(\varepsilon)}} A\left(\frac{D^\alpha f_{m\varepsilon}(y)}{2}\right) dy \leqq \int_Q A\left(\frac{D^\alpha f_m(y)}{2}\right) dy$$

and from our choice of Q we get: $\lim_{\varepsilon\to 0} I_2 = 0$.

Therefore, we can choose subsequences $m_k \to \infty$ and $\varepsilon_k \to 0$ such that by passing to the limit in (3.5) we get (3.4).

Now, for $x, x' \in R^n$ such that $|x - x'| = R < 1$, we take a ball $B(x_0, R)$ which contains them and from (3.4) we get:

$$|f(x) - f(x')| \leqq C_1 \sum_{|\alpha|=k} \left(\int_{B(x_0,R)} A(D^\alpha f(y))\, dy \right)^{k/n} \tag{3.6}$$

$$\leqq C_1 \sum_{|\alpha|=k} \left(\int_{B(x,2R)} A(D^\alpha f(y))\, dy \right)^{k/n}.$$

The rest of the proof follows from the fact that in the same way as Calderón showed it for $|\alpha| = 1$ in [1], we have for almost every x:

$$\lim_{R \to 0} \frac{1}{m(B(x,2R))} \int_{B(x,2R)} A(D^\alpha f(y) - D^\alpha f(x))\, dy = 0, \qquad |\alpha| = k.$$

So that by defining:

$$g(y) = f(y) - f(x) - \sum_{0<|\alpha|\leqq k} \frac{(D^\alpha f)(x)(y-x)^\alpha}{\alpha!}$$

and using (3.6) for g we get:

$$|g(x')| = |g(x) - g(x')| \leqq c_1 \sum_{|\alpha|=k} \left(\int_{B(x,2R)} A(D^\alpha f(y) - D^\alpha f(x))\, dy \right)^{k/n}$$

$$\leqq c_2 R^k \sum_{|\alpha|=k} \left(\frac{1}{m(B(x,2R))} \int_{B(x,2R)} A(D^\alpha f(y) - D^\alpha f(x))\, dy \right)^{k/n}.$$

Thus

$$\frac{|g(x')|}{|x - x'|^k} \to 0 \quad \text{as} \quad x' \to x$$

which means that f has a kth order differential at x. Q.E.D.

From Theorems 1 and 2 we get the following result:

Corollary 1. *If Ω and A are as in Theorem* 1 *and if $k < n$ then each equivalence class in $W^k L_A(\Omega)$ has a representative f which is differentiable of order k* a.e. *in Ω.*

Proof. We shall show that for an N-function A:

$$\int_0^1 \bar{A}(t^{k-n}) t^{n-1}\, dt < \infty \Leftrightarrow (3.3) \int_1^\infty \left[\frac{t}{A(t)} \right]^{k/(n-k)} dt < \infty \tag{1.1}$$

and then the corollary follows by taking the continuous representative of Theorem 1 and applying Theorem 2 on f.

First, we assume that A satisfies (1.1). From (2.5) it follows that

$$(3.7)\qquad \int_0^1 \bar{A}(t^{k-n})t^{n-1}\,dt = c\cdot\int_{\bar{A}(1)}^{\infty} s\cdot d([\bar{A}^{-1}(s)]^{n/(k-n)}).$$

Now, if f is a positive and nonincreasing function and if

$$\lim_{s\to\infty} f(s) = 0$$

the convergence of the integral $\int_{s_0}^{\infty} s\,df(s)$ implies that $\lim_{s\to\infty} sf(s) = 0$. This can be proved in the same way as Abel's theorem.

Therefore, we may integrate by parts the second integral in (3.7) and thus conclude that (1.1) implies

$$(3.8)\qquad \int_{\bar{A}(1)}^{\infty} [\bar{A}^{-1}(s)]^{n/(k-n)}\,ds < \infty.$$

Recalling Young's inequality, we get

$$(3.9)\qquad \int_{\bar{A}(1)}^{\infty} \left[\frac{A^{-1}(s)}{s}\right]^{n/(n-k)} ds < \infty.$$

Using Abel's theorem in (3.9) it follows that

$$(3.10)\qquad \lim_{s\to\infty} s\cdot\left[\frac{A^{-1}(s)}{s}\right]^{n/(n-k)} = \lim_{s\to\infty}\left[\frac{A^{-1}(s)}{s}\right]^{k/(n-k)} A^{-1}(s) = 0.$$

We have used here the fact that $t/A(t)$ is a decreasing function of t ([5, p. 8]).

Thus,

$$(3.11)\qquad \int_1^{\infty}\left[\frac{t}{A(t)}\right]^{k/(n-k)} dt = \int_{A(1)}^{\infty}\left[\frac{A^{-1}(s)}{s}\right]^{k/(n-k)} d(A^{-1}(s))$$

$$= \frac{n-k}{n}\left[\frac{A^{-1}(s)}{s}\right]^{k/(n-k)} A^{-1}(s)\Bigg]_{A(1)}^{\infty} + \frac{k}{n}\int_{A(1)}^{\infty}\left[\frac{A^{-1}(s)}{s}\right]^{n/(n-k)} ds.$$

From (3.9), (3.10) and (3.11) we obtain (3.3). We have thus shown that (1.1) implies (3.3).

Now, if we assume that A satisfies (3.3) from (3.11) we get (3.9) or equivalently (3.8). (1.1) follows in view of Abel's theorem from (3.7) and (3.8).

4.

Rozenfel'd has constructed an example of a function showing that Theorem 1 is sharp for $k = 1$. We shall now construct a function showing that condition (1.1) is sharp for all k. Rozenfel'd's construction in [8] cannot be generalized to the case $k > 1$; however Lemma 4.1, which describes the first step of the construction, is a lemma of Rozenfel'd.

LEMMA 4.1. *Let A be an N-function for which*

$$\int_0^1 \bar{A}(t^{k-n})t^{n-1}\,dt = \infty. \tag{4.1}$$

Then for each $\alpha > 0$, $\varepsilon > 0$ and $\rho > 0$ there exists a nonincreasing and nonnegative function $g_{\alpha,\varepsilon,\rho}(t) \in C^k$ such that $g_{\alpha,\varepsilon,\rho}(0) = \alpha$; $g^{(i)}_{\alpha,\varepsilon,\rho}(\rho) = 0$, $0 \leqq i \leqq k-1$ and

$$\int_0^\rho A\,(g^{(k)}_{\alpha,\varepsilon,\rho}(t))\,t^{n-1}\,dt < \infty.$$

PROOF. First, it is enough to prove the lemma for $\rho = 1$. Next we define

$$S(\alpha) = \inf_{g \in T_{|\alpha|}} \int_0^1 A\,(g^{(k)}(t))\,t^{n-1}\,dt$$

where T_α is the set of all nonincreasing and nonnegative $g \in C^k$ such that

$$g(0) = \alpha; \quad g^{(i)}(1) = 0, \quad 0 \leqq i \leqq k-1.$$

$S(\alpha)$ is a convex function. By using an argument similar to the one in [8] and the fact that for $g \in T_{|\alpha|}$ we have

$$\int_0^1 t^{k-1} g^{(k)}(t)\,dt = (-1)^k (k-1)!\,\alpha$$

it can be shown that the conjugate function is:

$$\bar{S}(p) = \begin{cases} 0 & p = 0 \\ \infty & p \neq 0. \end{cases}$$

This implies that $S(\alpha) = \bar{\bar{S}}(\alpha) \equiv 0$ as required.

REMARK 4.1. The function $g_{\alpha,\varepsilon,\rho}(t)$ can be extended to the interval $[0,1]$ as zero for $\rho < t \leqq 1$. Denoting by $\tilde{g}_{\alpha,\varepsilon,\rho}(t)$ the extension we have $\tilde{g}_{\alpha,\varepsilon,\rho} \in C^{k-1}(0,1)$,

$$\frac{d^{k-1}}{dt^{k-1}}\,\tilde{g}_{\alpha,\varepsilon,\rho} \quad \text{is Lipshitz}$$

$$\tilde{g}_{\alpha,\varepsilon,\rho}(0) = \alpha, \;\; \tilde{g}_{\alpha,\varepsilon,\rho}(t) = 0 \quad \text{for} \quad t \geqq \rho$$

and

$$\int_0^1 A\,(\tilde{g}^{(k)}_{\alpha,\varepsilon,\rho}(t))\,t^{n-1}\,dt < \varepsilon.$$

LEMMA 4.2. *Let $g \in C^k(0,1)$ be such that*

$$g(1) = g'(1) = \cdots = g^{(k-1)}(1) = 0.$$

Then for $\nu = 0, 1, \cdots, k-1$ *and the* N*-function* A:

$$\int_0^1 A(g^{(\nu)}(t))t^{n-1}dt \leqq \frac{1}{(n-1)^{k-\nu}}\int_0^1 A(g^{(k)}(t))t^{n-1}dt. \tag{4.2}$$

PROOF. It is enough to prove the lemma for $k = 1$. By hypothesis

$$g(t) = \int_1^t g'(s)ds.$$

From the convexity of A it follows that for $0 < t < 1$

$$A(g(t)) \leqq (1-t)A\left(\frac{g(t)}{1-t}\right) \leqq \int_t^1 A(g'(s))ds \leqq \frac{1}{t}\int_t^1 A(g'(s))ds.$$

Thus

$$\int_0^1 A(g(t))t^{n-1}dt \leqq \int_0^1\int_t^1 A(g'(s))ds\, t^{n-2}dt$$

$$= \int_0^1\int_0^s t^{n-2}dt\, A(g'(s))ds = \frac{1}{n-1}\int_0^1 A(g'(s))s^{n-1}ds.$$

Q.E.D.

Let us denote by $E_A(\Omega)$ the closure of the bounded functions in $L_A(\Omega)$ induced by the norm topology of $L_A(\Omega)$. (See [5] for the basic properties of $E_A(\Omega)$.) The Orlicz–Sobolev space $W^kE_A(\Omega)$ is defined in the same way as $W^kL_A(\Omega)$ with L_A replaced by E_A.

In the following, we shall need a theorem concerning the extension to R^n of functions belonging to an Orlicz–Sobolev space in a specific domain B. We shall bring here Stein's version of the extension theorem [10, p. 181, theor. 5]. Stein's theorem deals with Sobolev spaces but its proof can easily be applied to Orlicz–Sobolev spaces of type W^kE_A. The result is restricted to W^kE_A because $C^\infty(\Omega)$ is dense in $W^kE_A(\Omega)$ [2, p. 56] but not in $W^kL_A(\Omega)$.

STEIN'S EXTENSION THEOREM. *Let* B *be a domain with a bounded Lipshitz boundary. Then for each* $f \in W^kE_A(B)$ *there exists an* $\hat{f} \in W^kE_A(R^n)$ *such that*

$$\hat{f}(x) = f(x) \quad \text{for} \quad x \in B$$
$$\|\hat{f}\|_{W^kL_A(R^n)} \leqq c\,\|f\|_{W^kL_A(B)}. \tag{4.2}$$

Moreover if f *is continuous and bounded in* B *then so will be* $\hat{f}$ *in* R^n *and*

$$\max_{x\in R^n}|\hat{f}(x)| \leqq \max_{x\in B}|f(x)|. \tag{4.3}$$

REMARK 4.2. If ∂B can be divided into N parts such that in each of them the boundary can be represented by a Lipshitzian function, then the constant c in (4.2) depends only on k, n, N and the maximum of the Lipshitz constants corresponding to the N parts.

THEOREM 3. *Let Ω be a domain in R^n. Then a necessary condition that $W^k L_A(\Omega)$ be embedded in $C(\Omega) \cap L^\infty(\Omega)$ is that*

$$\int_0^1 \bar{A}(t^{k-n}) t^{n-1} dt < \infty .$$

PROOF. We shall show that if A satisfies (4.1) then there exists a function belonging to $W^k L_A(R^n)$ which is unbounded in every neighborhood of the origin.

Let $h_i(t) = \tilde{g}_{i,1/(k+1),1/i}(t)$ where $\tilde{g}_{\alpha,\varepsilon,\rho}$ are obtained from Remark 4.1. Let

$$B = \{(x_1, \cdots, x_n) \mid 0 < x_2, \cdots, x_n < x_1, 0 < x_1 < 1\} \tag{4.4}$$

and for $x \in B$ let us consider

$$F_i(x_1, \cdots, x_n) = h_i(x_1).$$

Then for $\nu = 0, 1, \cdots, k-1$

$$\begin{aligned} \sum_{|\alpha|=\nu} \int_B A(D^\alpha F_i(x))dx &= \int_0^1 \int_0^{x_1} \cdots \int_0^{x_1} A(h_i^{(\nu)}(x_1)) dx_2 \cdots dx_n dx_1 \\ &= \int_0^1 A(h_i^{(\nu)}(x_1)) x_1^{n-1} dx_1 \leqq \frac{1}{(n-1)^{k-\nu}} \int_0^1 A(h_i^{(k)}(t)) t^{n-1} dt < \frac{1}{k+1}, \end{aligned} \tag{4.5}$$

where we have used Lemma 4.2. Thus we get

$$\| F_i \|_{W^k L_A(B)} = \sum_{|\alpha| \leqq k} \| D^\alpha F_i \|_A = \sum_{\nu=0}^{k} \sum_{|\alpha|=\nu} \| D^\alpha F_i \|_A < 1 .$$

Furthermore, since the derivatives of h_i are bounded, $F_i \in W^k E_A(B)$. Now let

$$f(x) = \sum_{i=1}^{\infty} \frac{F_i(x)}{i^2} .$$

Since $F_i \geqq 0$ is continuous at the origin and $F_i(0) = i$, obviously

$$\lim_{x \to 0} f(x) = \infty .$$

On the other hand, f is the limit of a Cauchy sequence in $W^k E_A(B)$ since

$$\left\| \sum_{i=m}^{l} \frac{F_i(x)}{i^2} \right\|_{W^k L_A(B)} \leq \sum_{i=m}^{l} \frac{1}{i^2} \to 0 .$$

Thus $f \in W^k E_A(B)$. Now, B is a Lipshitz domain and from Stein's extension theorem f can be extended to all of R^n such that $\hat{f} \in W^k L_A(R^n)$ and $\hat{f}$ is discontinuous and unbounded in the origin. Q.E.D.

The following theorem shows that condition (3.3) of Theorem 2 is sharp.

THEOREM 4. *If*

$$\int_1^\infty \left[\frac{t}{A(t)} \right]^{k/(n-k)} dt = \infty \tag{4.6}$$

then there exists a continuous function in $W^k L_A(R^n)$ which is not differentiable (and therefore not differentiable of order k) on a set of positive measure.

PROOF. From the equivalence of (1.1) and (3.3) (see proof of Corollary 1) it follows that (4.6) implies (4.1). We can thus consider the functions

$$h_i(t) = \tilde{g}_{4^{in}(2/3)^i,\, 1/(k+1),\, 1/4^i\sqrt{n}}(t)$$

where $\tilde{g}_{\alpha,\varepsilon,\rho}(t)$ are the functions we get from Remark 4.1.

Let B be defined by (4.4) and let us consider the sets

$$K_i = \left\{ (x_1, \cdots, x_n) \,|\, -\frac{1}{4^i\sqrt{n}} \leq x_j \leq \frac{1}{4^i\sqrt{n}},\ 1 \leq j \leq n \right\}$$

$$B_i = \left\{ (x_1, \cdots, x_n) \,|\, x \in B,\ x_1 < \frac{1}{4^i\sqrt{n}} \right\}$$

$$D_i = R^n \setminus (K_i \setminus B_i) .$$

For $x \in D_i$ we define the function

$$F_i(x_1, \cdots, x_n) = \begin{cases} h_i(x_1) & x \in B_i \\ 0 & x \in R^n \setminus K_i . \end{cases}$$

Then $F_i \geqq 0$, $\lim_{x \to 0} F_i(x) = 4^{in}(2/3)^i$ and, for $|x| \geqq 1/4^i$, $F_i(x) = 0$. Now, by making use of Lemma 4.2 and (4.5) we get

$$\| F_i \|_{W^k L_A(D_i)} = \sum_{|\alpha| \leqq k} \| F_i \|_{L_A(D_i)} < 1 .$$

Moreover, by definition F_i and its derivatives of order less than or equal to k are bounded which implies that $F_i \in W^k E_A(D_i)$. Therefore, by making use of Stein's extension theorem, F_i can be extended to R^n. Thus we get the continuous functions $\hat{F}_i \in W^k E_A(R^n)$ such that for each i

$$\| \hat{F}_i \|_{W^k L_A(R^n)} \leqq c \| F_i \|_{W^k L_A(D_i)} \leqq c.$$

The constant c is independent of i because the constant in Stein's theorem depends on the upper bound of the Lipshitz constants of the domain D_i and these constants are independent of i.

Furthermore,

$$\hat{F}_i(0) = 4^{in} (\tfrac{2}{3})^i$$

and from (4.3) we get that

$$| \hat{F}_i(x) | \leqq 4^{in} (\tfrac{2}{3})^i.$$

Now, let x_h^i be the interior points of B such that their coordinates are integral multiples of $1/2^i$ and let n_i be the number of such points in B. Clearly $n_i < 2^{in}$. We define

$$G_i(x) = \sum_{h=1}^{n_i} \hat{F}_i(x - x_h^i).$$

Then

$$\| G_i \|_{W^k L_A} \leqq n_i \| \hat{F}_i \|_{W^k L_A} \leqq 2^{in} c. \tag{4.7}$$

And by definition

$$| x - x_h^i | \geqq \frac{1}{4^i} \Rightarrow \hat{F}_i(x - x_h^i) = 0.$$

Therefore for each x, at most one of the terms in the sum defining $G_i(x)$ is different from zero, and

$$| G_i(x) | \leqq 4^{in} (\tfrac{2}{3})^i.$$

Now let

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{4^{in}} G_i(x).$$

First, the series defining f converges uniformly in R^n and therefore f is continuous there. Secondly, $f \in W^k L_A(R^n)$. Indeed, using (4.7) we have

$$\lim_{l,m \to \infty} \sum_{l}^{m} \frac{1}{4^{in}} \| G_i \|_{W^k L_A(R_n)} \to 0.$$

Finally, as it is shown in [1, p. 211], f is not differentiable on a set of positive measure and therefore not differentiable of order k on that set. Q.E.D.

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